

Finite size dependence of scaling functions of the three dimensional $O(4)$ model in an external field

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Abstract

We calculate universal finite size scaling functions for the order parameter and the longitudinal susceptibility of the three-dimensional $O(4)$ model. The phase transition of this model is supposed to be in the same universality class as the chiral transition of two-flavor QCD. The scaling functions serve as a testing device for QCD simulations on small lattices, where, for example, pseudocritical temperatures are difficult to determine. In addition, we have improved the infinite volume limit parametrization of the scaling functions by using newly generated high statistics data for the $3d$ $O(4)$ model in the high temperature region on an $L = 120$ lattice.

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1 Introduction

The three-dimensional $O(4)$ spin model plays an important role for our understanding of the low energy limit of Quantumchromodynamics (QCD) as well as its phase structure at non-zero temperature. The almost massless pseudo-scalar particles (pions) of QCD have been considered already early as Goldstone particles, which receive a non-zero mass only due to the presence of a chiral symmetry breaking field, i.e. the non-zero light quark masses. The consequences of this idea for the QCD phase diagram have been worked out in detail in a paper by Pisarski and Wilczek [1]. Their conclusion is that, if QCD undergoes a second-order chiral transition in the limit of vanishing up and down quark masses, this transition is expected to belong to the universality class of the three-dimensional $O(4)$ spin model ¹. Thermodynamics in the vicinity of the QCD transition will then show universal features that are described by $O(4)$ scaling functions. In fact, recent studies of the quark mass dependence of the chiral transition in lattice QCD using the staggered fermion discretization scheme led to good agreement with infinite volume $O(2)$ or $O(4)$ scaling functions [2].

Lattice QCD calculations with light dynamical quarks are performed in relatively small volumes. This is in particular the case for the computationally demanding chiral fermion formulations, i.e., for studies of QCD thermodynamics performed with domain wall fermions [3] or overlap fermions [4]. A finite volume limits the correlation length and thus, similar to the influence of an external field, it modifies the universal behavior in the vicinity of a second order phase transition. It has been shown that the finite volume dependence of the chiral condensate close to the QCD transition temperature may be understood in terms of finite volume scaling functions of the $3d$ $O(4)$ spin model [5].

The universal properties of the $O(4)$ model in the vicinity of the symmetry restoring phase transition are characterized by critical exponents, and universal scaling functions. Quite generically a second order phase transition occurs at a certain temperature value, T_c , only in the thermodynamic limit, $1/L \equiv 1/V^{1/3} = 0$, and at vanishing external field, $H = 0$. The approach to the critical point is then usually controlled by three parameters $h = H/H_0$, $l = L_0/L$ and $\bar{t} = (T - T_c)/T_0$. The $O(4)$ scaling functions, f_G and f_χ , that describe the universal behavior of the order parameter and its susceptibility in the vicinity of the critical point, have been analyzed in quite some detail in the thermodynamic limit. These functions depend on the scaling variable $z = \bar{t}/h^{1/\Delta}$, where $\Delta = \beta\delta$ is the so-called gap exponent. Recently, also the scaling function $f_f(z)$ for the singular part of the free energy density has been determined [6]. The functions f_G and f_χ are obtained from $f_f(z)$ by taking appropriate derivatives with respect to H . While finite size scaling

¹This is not settled with certainty as the influence of the axial anomaly on the transition is still not fully understood. The effective restoration of $U_A(1)$ at high temperature may indeed trigger a first order transition.

relations are frequently discussed in the absence of an external field, much less is known about finite volume effects in the presence of such a field [5].

Calculations performed within the framework of an $O(4)$ symmetric quark-meson model, using an approximation scheme based on the functional renormalization group (FRG) approach [7, 8], suggest as well that finite size effects may influence the determination of the chiral transition temperature, or more generally, the temperature dependence of the chiral susceptibility in QCD. It thus is of interest to arrive at quantitative results for the finite size dependence of the scaling functions of the $3d$ $O(4)$ universality class which can be used in scaling studies of thermodynamic observables determined in lattice QCD. Within the FRG approach [9] these scaling functions have been derived approximately (In Ref. [9] the exponent η is zero, unlike in the $O(4)$ class.). In this paper we will determine finite volume $O(4)$ scaling functions of the order parameter and the chiral susceptibility directly from high statistics Monte Carlo simulations of the three-dimensional $O(4)$ spin model.

Our paper is organized as follows. In section II we extend the relations for the infinite volume scaling functions to the case of finite $L = V^{1/d}$. The simulations of the $3d$ $O(4)$ model on lattices with $L = 24 - 120$ and the resulting finite volume dependencies of the scaling functions f_G and f_χ are then discussed in section III. In addition we improve the parametrization of the infinite volume scaling functions as given in Ref. [6]. We close with a summary and the conclusions.

2 Finite size dependence of scaling functions

In order to introduce the finite size dependence of the scaling functions we reconsider parts of chapter 2 of Ref. [6]. The scaling functions are derived from the reduced (containing a factor $\beta = 1/T$) free energy density,

$$f = -\frac{1}{V} \ln Z(T, H, L) = f_s(T, H, L) + f_{ns}(T, H, L) . \quad (1)$$

Here, we have split the free energy density in a singular term f_s , responsible according to renormalization group (RG) theory for critical behavior, and a regular or non-singular term f_{ns} . The scaling laws near the critical point are derived from the RG scaling equation for f_s . The derivatives of f_{ns} contribute regular terms to the scaling laws, which apart from the cases of the energy density and the specific heat (for $\alpha < 0$) are sub-leading near the critical point. The dependence on L or better on $1/L$ can be treated as a correction to leading scaling behavior, that is $l = L_0/L$ takes the rôle of an additional relevant scaling field with exponent $y_L = 1$. If we neglect the contributions of all irrelevant scaling fields close to the critical point, because of their negative exponents y_j , then the RG equation becomes

$$f_s(u_t, u_h, l) = b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h, b l) . \quad (2)$$

Here, b is a free positive scale factor and the usual, remaining relevant scaling fields are $u_t = c_t t$, $u_h = c_h H$, where $t = (T - T_c)/T_c$. The c_t, c_h and L_0 are model-dependent (positive) metric scale factors. By choosing $b = u_h^{-1/y_h}$ for $H > 0$ one obtains from Eq. (2) the form of scaling functions which we want to discuss here

$$f_s(u_t, u_h, l) = u_h^{d/y_h} f_s(u_t u_h^{-y_t/y_h}, 1, l u_h^{-1/y_h}) . \quad (3)$$

The singular term f_s is a universal function of the scaling fields u_t, u_h and l and

$$f_s = (c_h H)^{d/y_h} \Psi_2(c_t c_h^{-y_t/y_h} t H^{-y_t/y_h}, c_h^{-1/y_h} l H^{-1/y_h}) , \quad (4)$$

where Ψ_2 is again a universal function but in contrast to the thermodynamic limit now of two arguments. By comparison with the infinite volume scaling laws one derives

$$y_t = 1/\nu , \quad y_h = 1/\nu_c = \Delta/\nu , \quad \text{or} \quad \Delta = y_h/y_t , \quad (5)$$

and the known hyperscaling relations between the critical exponents.

Instead of working with the metric scale factors c_t and c_h one introduces usually new temperature and field variables $\bar{t} = tT_c/T_0$ and $h = H/H_0$ in the thermodynamic limit. We stick to this tradition also in the finite volume case. Correspondingly, the scaling functions of the observables which are derivatives of the free energy density will depend (see Eq. (4)) on the two variables

$$z = \bar{t}/h^{1/\Delta} , \quad \text{and} \quad z_L = l/h^{\nu_c} , \quad (6)$$

and the thermodynamic limit is recovered for $z_L = 0$. For example, the order parameter, or magnetization becomes then

$$M = -\frac{\partial f}{\partial H} = h^{1/\delta} f_G(z, z_L) . \quad (7)$$

In order to specify the scaling variables z and z_L in a given model calculation one needs to fix the non-universal scales T_0 , H_0 and L_0 . As already mentioned, the first two are fixed by demanding in the infinite volume limit

$$M(t=0) = h^{1/\delta} \quad \text{and} \quad M(h=0) = (-t)^\beta . \quad (8)$$

This implies

$$f_G(0,0) = 1 , \quad \text{and} \quad f_G(z,0) \xrightarrow{z \rightarrow -\infty} (-z)^\beta . \quad (9)$$

The scale L_0 is fixed by a third normalization condition. We choose

$$z_L = 1 \quad \text{for} \quad \frac{M(t=0)|_L}{M(t=0)|_{L=\infty}} = f_G(0,1) = 0.874 . \quad (10)$$

That is, on a given lattice of size $V = L^d$ we determine at $t = 0$ the finite volume scaling function $f_G(0, z_L)$ by varying the external field h . We then assign the value $z_L = 1$ to that value of h that yields $f_G(0,1) = 0.874$. This fixes the scale L_0

to be $L_0 = Lh^{\nu_c}$. Our normalization condition seems arbitrary. In fact, we could have chosen other normalization conditions. However, in the absence of any obvious natural choice the above normalization condition is convenient. As we shall see this condition leads to $L_0 \equiv 1$ for the 3d $O(4)$ spin model studied here.

Due to Eq. (4) and the derivative in Eq. (7) the singular term f_s depends on the scaling function $f_f(z, z_L)$

$$f_s = H_0 h^{1+1/\delta} f_f(z, z_L) , \quad (11)$$

which relates to f_G by

$$f_G(z, z_L) = - \left(1 + \frac{1}{\delta} \right) f_f(z, z_L) + \frac{z}{\Delta} \frac{\partial f_f(z, z_L)}{\partial z} + \nu_c z_L \frac{\partial f_f(z, z_L)}{\partial z_L} . \quad (12)$$

Fluctuations of the order parameter in the $O(4)$ model, are described by the longitudinal susceptibility,

$$\chi_L = \frac{\partial M}{\partial H} = \frac{h^{1/\delta-1}}{H_0} f_\chi(z, z_L) , \quad (13)$$

with

$$f_\chi(z, z_L) = \frac{1}{\delta} f_G(z, z_L) - \frac{z}{\Delta} \frac{\partial f_G(z, z_L)}{\partial z} - \nu_c z_L \frac{\partial f_G(z, z_L)}{\partial z_L} . \quad (14)$$

A further observable of interest is the thermal susceptibility χ_t , the mixed second derivative of f

$$\chi_t = \frac{\partial M}{\partial(1/T)} = - \frac{T^2}{T_0} h^{(\beta-1)/\Delta} \frac{\partial f_G(z, z_L)}{\partial z} . \quad (15)$$

Both χ_L and χ_t have their counterparts in QCD.

We note that our finite size scaling functions relate in a simple way to the ones that are commonly used in the literature [5, 9]. In terms of these functions our observables read

$$M = L^{-\beta/\nu} Q_G(z, z_L) , \quad (16)$$

$$\chi_L = \frac{L^{\gamma/\nu}}{H_0} Q_\chi(z, z_L) , \quad (17)$$

$$\chi_t = \frac{T^2}{T_0} L^{-(\beta-1)/\nu} Q_t(z, z_L) , \quad (18)$$

and the connection to our scaling functions is

$$Q_G(z, z_L) = (z_L/L_0)^{-\beta/\nu} f_G(z, z_L) , \quad (19)$$

$$Q_\chi(z, z_L) = (z_L/L_0)^{\gamma/\nu} f_\chi(z, z_L) , \quad (20)$$

$$Q_t(z, z_L) = (z_L/L_0)^{-(\beta-1)/\nu} \left(- \frac{\partial f_G(z, z_L)}{\partial z} \right) . \quad (21)$$

Of particular interest for the determination of the infinite volume scaling functions is the leading (in L) or asymptotic form of the Q -functions [5]. One obtains for $L \rightarrow \infty$

$$Q_G \rightarrow f_G(z, 0)(hL^{1/\nu_c})^{1/\delta}, \quad Q_\chi \rightarrow f_\chi(z, 0)(hL^{1/\nu_c})^{1/\delta-1}, \quad (22)$$

$$Q_t \rightarrow -\frac{\partial f_G(z, 0)}{\partial z}(hL^{1/\nu_c})^{(\beta-1)/\Delta}, \quad (23)$$

that is, at fixed z we have powers of hL^{1/ν_c} with infinite volume scaling functions as coefficients.

3 Simulation of the 3d $O(4)$ model

We determine the finite size scaling functions from simulations of the standard $O(4)$ -invariant nonlinear σ -model. It is defined by

$$\beta \mathcal{H} = -J \sum_{\langle \vec{x}, \vec{y} \rangle} \vec{\phi}_{\vec{x}} \cdot \vec{\phi}_{\vec{y}} - \vec{H} \cdot \sum_{\vec{x}} \vec{\phi}_{\vec{x}}, \quad (24)$$

where \vec{x} and \vec{y} are nearest-neighbor sites on a three-dimensional hyper-cubic lattice, $\vec{\phi}_{\vec{x}}$ is a four-component unit vector at site \vec{x} . The coupling J and the external field \vec{H} are reduced quantities, that is they contain already a factor $\beta = 1/T$. That allows us to consider the coupling directly as the inverse temperature, $J \equiv 1/T$. An additional factor on J would just change the scale of T . The setup of our calculations and the definition of further observables are given in detail in Ref. [6]. There are two differences to the simulations in Ref. [10], which we used in [6] to calculate the infinite volume volume scaling functions $f(z) \equiv f(z, 0)$. Instead of calculating at fixed T and varying H , we fix z and vary H and secondly we use, according to the average clustersize, an appropriate number of cluster updates such as to cover before each measurement the whole volume V of the lattice 5 to 6 times. That becomes relevant in particular with increasing $z > 0$ and leads there to much better statistics. For example, on a lattice of size $L = 120$ we made between 400 (at $z = 0.1$) and 14.000 cluster updates (at $z = 4.0$) for each of our 100.000 measurements. In order to define our variables t, \bar{t}, h and z we use the same critical amplitudes, temperature and exponent values for the 3d $O(4)$ model as in Refs. [10] and [11]. These are

$$J_c = T_c^{-1} = 0.93590, \quad T_0 = 1.093, \quad H_0 = 4.845, \quad \beta = 0.380, \quad \delta = 4.824. \quad (25)$$

From the hyperscaling relations one obtains then

$$\alpha = -0.2131, \quad \nu = 0.7377, \quad \gamma = 1.4531, \quad \Delta = 1.83312, \quad \nu_c = 0.40243. \quad (26)$$

The main goal of our simulations was, as already explained, to study the finite size dependence of the scaling functions f_G and f_χ . We have done this by calculating

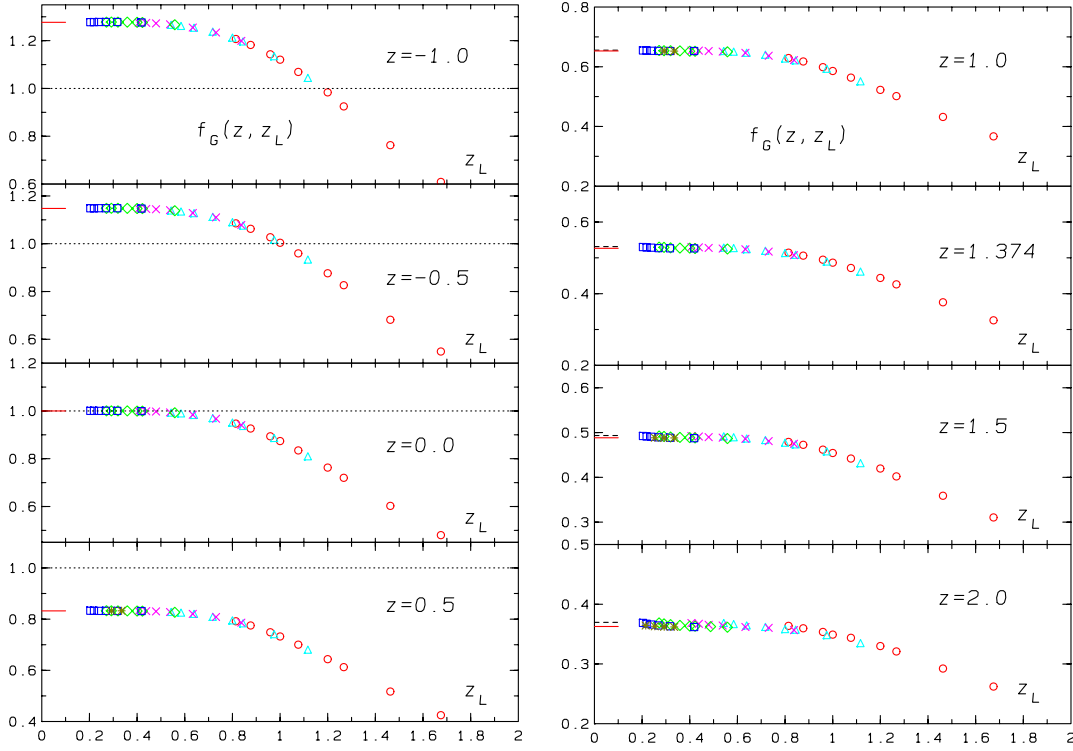


Figure 1: The finite volume scaling function $f_G(z, z_L)$ for some values of z versus z_L . For each value of z lattices of size $L = 24$ (circles), 36 (triangles), 48 (crosses), 72 (diamonds), 96 (squares) and various values of h have been used. For $z = 0.5, 1.0, 1.5$ and 2.0 also data for $L = 120$ (stars) have been added. Short horizontal lines close to $z_L = 0$ show the values of the infinite volume scaling function $f_G(z)$, the solid lines denote the new parametrization, the dashed lines the one of Ref. [6].

the scaling functions at fixed z and varying $z_L = l/h^{\nu_c}$. Here, lattices of size $L = 24, 36, 48, 72, 96$ and external fields H in the range $[0.0005, 0.003]$ have been used, supplemented at some z -values by results from $L = 120$ lattices. The z -range which we have considered is $[-1.0, 2.0]$, that is, we cover the region around the critical point and the high temperature region up to and including the peak area of $f_\chi(z, 0)$. In Ref. [6] we had determined the peak position to $z_p = 1.374(30)$. It leads to the pseudocritical line $z = z_p$ in the (t, h) -plane where χ_L is at its maximum for fixed h and varying t .

All our results for the scaling function $f_G(z, z_L)$ are summarized in Fig. 1. In order to plot the data as a function of $z_L = L_0/Lh^{\nu_c}$ we have fixed $L_0 = 1$. As can be seen in the $(z = 0)$ -part of the plot that amounts to the normalization condition Eq. (10). In Fig. 2 we have used the same data to show the z -dependence of $f_G(z, z_L)$ at fixed z_L -values. We see from the two figures that finite size effects in the scaling function f_G are small for $z_L \leq 0.5$. The main effects appear in the low temperature

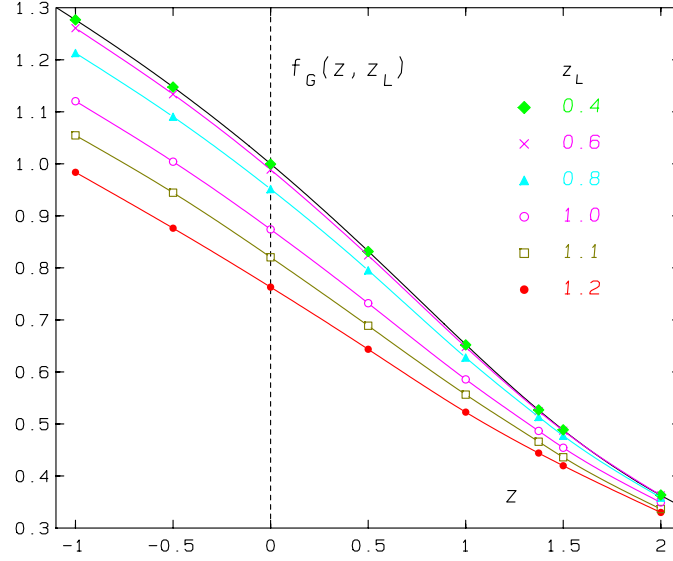


Figure 2: The finite volume scaling function $f_G(z, z_L)$ for some values of z_L versus z . Curves shown are spline interpolations of the data, the points for $z_L = 0.4$ are not connected, the highest line shows $f_G(z, 0)$ from the new parametrization.

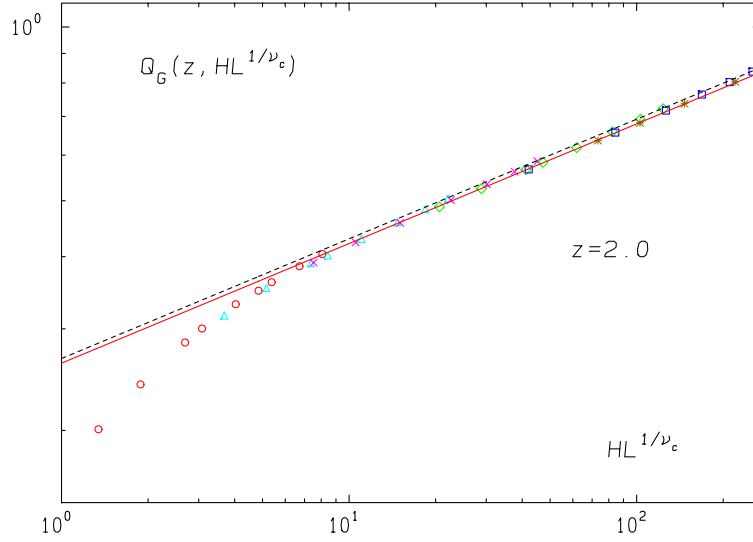


Figure 3: The finite volume scaling function Q_G for $z = 2$ versus HL^{1/ν_c} . The lines show the asymptotic forms, Eq. (22), calculated from $f_G(z)$ using the parametrizations of Ref. [6] (dashed line) and the one given in Appendix A (solid line).

region and decrease with increasing z . This is in particular evident from Fig. 2.

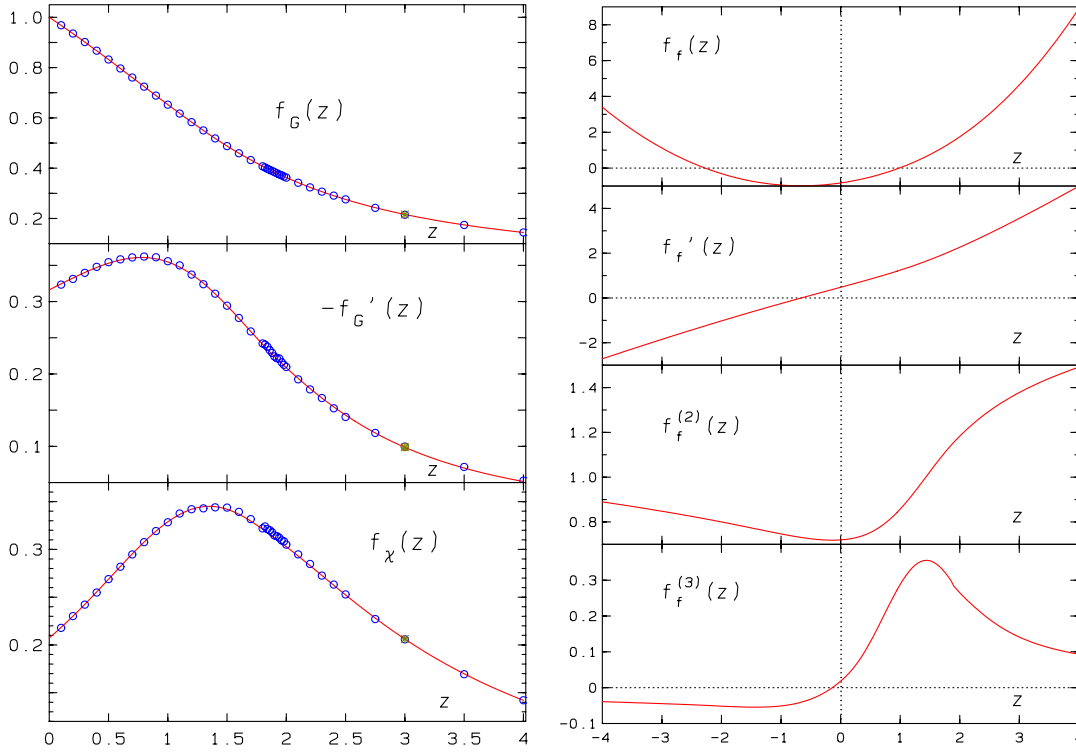


Figure 4: Left part: The infinite volume scaling functions $f_G(z)$, $-f'_G(z)$ and $f_\chi(z)$ for $z > 0$, $T > T_c$ from data (circles) for M , χ_t and χ_L on an $L = 120$ lattice. The lines are from the parametrization of the scaling functions in Appendix A. Right part: The scaling function $f_f(z)$ and its first three derivatives from the new parametrization.

Another important observation is that we find perfect finite size scaling in z_L for $z \leq 1.0$. At larger values of z , however, the results from different size lattices start to deviate from each other. In addition, the data at fixed z do no longer reach a plateau for $z_L \rightarrow 0$, even for the same lattice size. Obviously, one needs here data on larger lattices and/or smaller H -values. The effect is known from Ref. [5], where the scaling functions Q_G had been calculated for $z = 0$ and $z = z_p$. In Fig. 4 of Ref. [5], $Q_G(z_p)$ is shown as a function of HL^{1/ν_c} , instead of z_L . One observes, that with increasing H the corresponding data at fixed L overshoot the asymptotic result and this happens the earlier the smaller L is. Only for small H and large L there is a small window, where we have coincidence with the asymptotic behavior expected from Eq. (22). In fact, also our parametrization from Ref. [6], which was based on results from lattices with $L = 120$, is apparently affected by the use of data with too large field values in the peak region. In order to check this we have made a logarithmic plot of the scaling function $Q_G = ML^{\beta/\nu}$ at $z = 2$ versus HL^{1/ν_c} with our new high statistics data. As is clearly seen in Fig. 3 the asymptotic form of Q_G predicted by the parametrization in Ref. [6] is slightly too high at $z = 2$. In order

to remedy the deficiencies we had had in parametrizing the infinite volume scaling functions for $z > 0$ in Ref. [6], namely too low statistics, too high field values and no systematic covering of the necessary z -range, we decided to produce new high statistics data on an $L = 120$ lattice at fixed z -values with two or three low field values. These new data are shown in Fig. 4 for $f_G(z)$, $-f'_G(z)$ and $f_\chi(z)$ together with a new parametrization, whose details we list in Appendix A. The new data cover now in particular the peaks with sufficient accuracy. In Figs. 1 and 2 we have already displayed the infinite volume values for f_G close to $z_L = 0$. We see that a difference to the old parametrization is here only visible for $z \gtrsim 1$. In the right part of Fig. 4 we show the scaling function of the free energy density and its first three derivatives. Our results for the finite volume scaling function of the susceptibility are shown in Figs. 5 and 6. We see again the strongest finite size effects for the smaller values of z . In particular, the peak of the susceptibility is washed out completely on small lattices or large z_L , so that there a pseudocritical temperature cannot be determined safely. We note, that the peak position of $f_\chi(z, 0)$ has shifted slightly to $z_p = 1.35(3)$ due to the new parametrization. The value is, however, inside the error bars of the old value $z_p = 1.374(30)$. In Figs. 2 and 6 we have shown spline interpolations of our data. When using the scaling functions in the analysis of data for other models, it may be more convenient to use polynomial interpolation formulas. We have fitted simultaneously all our data for $f_G(z, z_L)$ and $f_\chi(z, z_L)$ to a polynomial ansatz for $f_G(z, z_L)$,

$$f_G(z, z_L) = f_G(z, 0) + \sum_{n=0}^4 \sum_{m=3}^7 a_{nm} z^n z_L^m. \quad (27)$$

We use here of course the new parametrization of the infinite volume scaling function $f_G(z, 0)$ as discussed in Appendix A. The ansatz for $f_\chi(z, z_L)$ is obtained from $f_G(z, z_L)$ using Eq. (14). We obtained the expansion coefficients a_{nm} from a simultaneous fit to data for $f_G(z, z_L)$ and $f_\chi(z, z_L)$. As can be seen in Figs. 1 and 5 in the interval $z \in [-1.0, 2.0]$ significant finite size effects in f_G and f_χ only set in for $z_L \geq 0.5$. For this reason we use in Eq. (27) a polynomial ansatz that starts with

a_{nm}		n				
		0	1	2	3	4
m	3	0.0421332	-0.0782771	0.0546495	-0.0251385	0.0017542
	4	0.0576183	0.3302893	-0.2642637	0.0617961	0.0049618
	5	-0.6352819	-0.3461722	0.4678005	-0.0453606	-0.0309722
	6	0.5355251	0.1770113	-0.3118316	0.0061252	0.0295072
	7	-0.1247180	-0.0369583	0.0696270	0.0021488	-0.0078913

Table 1: Expansion coefficients

finite volume corrections at $\mathcal{O}(z_L^3)$. That is, the leading corrections are proportional to $1/L^3$. We also performed fits with smaller powers of z_L which, however, did not improve over the current ansatz. Results for the expansion coefficients a_{nm} are given in Table 1. This parametrization is appropriate for f_G and f_χ in the intervals $z \in [-1.0, 2.0]$ and $z_L \in [0.0, 1.2]$.

4 Summary and conclusions

In our paper we have investigated the finite size scaling functions of the universality class of the $3d$ $O(4)$ spin model. Our aim was to provide a suitable form for tests on this universality class of QCD data, which were obtained from simulations on small lattices. In contrast to the commonly studied finite size scaling functions Q_G [5], Q_χ and so forth, we have directly extended the infinite volume scaling functions to finite size scaling functions $f(z, z_L)$, which describe the size dependence as well. The z -region, where we have examined these functions, encompasses the vicinity of the critical point and the high temperature region up to and including the peak area of $f_\chi(z, 0)$. Thereby, we cover the domain where the pseudocritical line obtained from the susceptibility is of interest. In order to actually use the finite size scaling functions for a test of a model, one has, of course, to determine the 4 model specific parameters T_c, T_0, H_0 and L_0 .

We have seen from Figs. 1 and 2 for $f_G(z, z_L)$ and Figs. 5 and 6 for $f_\chi(z, z_L)$ that on one hand finite size effects are small for $z_L < 0.5$ and on the other hand that the main finite size effects appear in the small z -region and that they decrease with increasing z . In the course of our analysis we found out that the parametrization of the infinite volume scaling function $f_G(z)$ as given in Ref. [6] had to be improved in the region $z \geq 1.0$. We have therefore generated new high statistics data on an $L = 120$ lattice at fixed z in the range $[0.1, 4.0]$ at several small field values each. These data enabled us to update the parametrization [6]. Its results are presented in Appendix A. The expansion coefficients given in Table 1 are based on the new parametrization of the infinite volume scaling function.

As a by-product of our calculations we obtained new values for the universal product $R_\chi = d_0^+ = 1.0919(14)$ and the universal ratio $A^+/A^- = 1.734(75)$. Also the peak positions of f_χ and $-f'_G$ changed slightly to $z_{p,\chi} = 1.35(3)$ and $z_{p,t} = 0.78(4)$, respectively.

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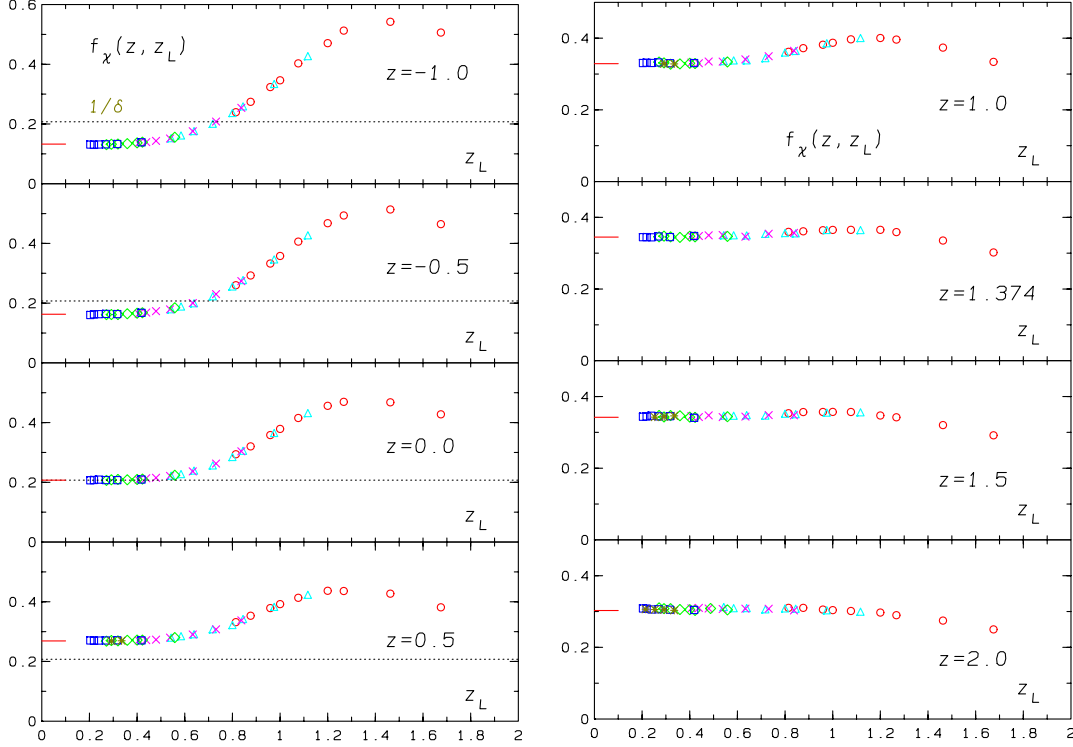


Figure 5: The finite volume scaling function $f_\chi(z, z_L)$ for some values of z versus z_L . For each value of z lattices of size $L = 24-96$ and various values of h have been used. For $z = 0.5, 1.0, 1.5$ and 2.0 we have also added results from lattices of size $L = 120$. The notation for the symbols is as in Fig. 1. The infinite volume predictions for f_χ are shown as short horizontal lines.

A Update on the infinite volume form of the scaling function f_G

As already shown in Section 3, Fig. 4, we have new data for $z > 0$. We have used these data and those of Ref. [6] for $z \leq 0$ to update the parametrization in [6] of the infinite volume scaling functions. We started with the Taylor series around $z = 0$

$$f_G(z) = \sum_{n=0}^{\infty} b_n z^n. \quad (28)$$

Like in [6], we have fitted the data from $-f_G(z)$ for small z . Here, this was done once in the z -range $[-2.5, 1.0]$ up to $n = 7$, and once in the range $[-0.5, 2.3]$ up to $n = 8$ but with the coefficients b_0, \dots, b_4 from the first fit as input. The results of the first fit are to be used for $z \leq 0$, the others for $z > 0$. The coincidence of the

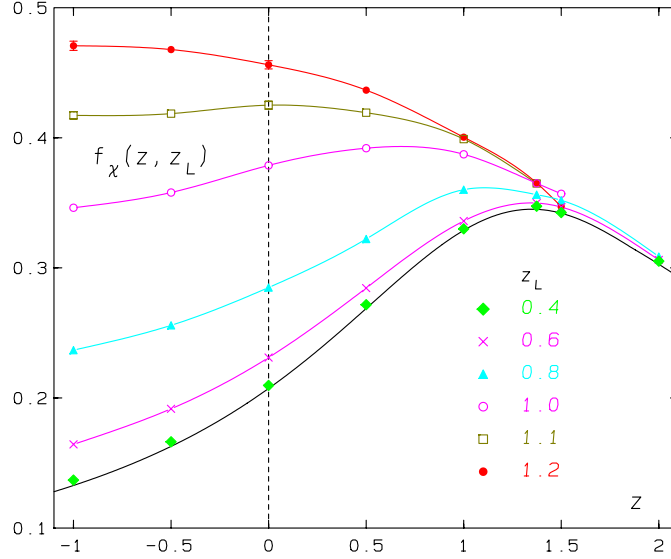


Figure 6: The finite volume scaling function $f_\chi(z, z_L)$ for some values of z_L versus z . Curves shown are spline interpolations of the data, the points for $z_L = 0.4$ are not connected, the lowest line shows $f_\chi(z, 0)$ from the new parametrization.

lowest coefficients guarantees smooth fits at $z = 0$ for all the derivatives we need. The final result for the coefficients is

$$b_0 \equiv 1, \quad b_1 = -0.316123 \pm 0.000628, \quad b_2 = -0.0418371 \pm 0.000661, \quad (29)$$

$$b_3 = 0.00129543 \pm 0.000859, \quad b_4 = 0.00582947 \pm 0.000345. \quad (30)$$

The remaining coefficients are different for $z < 0$ and $z > 0$. We find for $z > 0$

$$b_5^+ = 0.00201377 \pm 0.000887, \quad b_6^+ = 0.0048998 \pm 0.001122, \quad (31)$$

$$b_7^+ = -0.00386218 \pm 0.000484, \quad b_8^+ = 0.00068511 \pm 0.000071, \quad (32)$$

and for $z < 0$

$$b_5^- = 0.00257848 \pm 0.000450, \quad b_6^- = 0.00053247 \pm 0.000211, \quad (33)$$

$$b_7^- = 0.000044801 \pm 0.000030. \quad (34)$$

We next consider the asymptotic expansions. In the high temperature region, that is for $z \rightarrow \infty$, or for $t > 0$ and $h \rightarrow 0$, we use the ansatz

$$f_G(z) = z^{-\gamma} \cdot \sum_{n=0}^{\infty} d_n^+ z^{-2n\Delta}. \quad (35)$$

In the low temperature region, for $t < 0$ and $h \rightarrow 0$, we make the following ansatz for $f_G(z)$

$$f_G(z) = (-z)^\beta \cdot \sum_{n=0}^{\infty} d_n^- (-z)^{-n\Delta/2} . \quad (36)$$

We have fitted f_G in the positive z -range $[1.5, 4]$ with the first 4 terms of Eq. (35) and found

$$d_0^+ = 1.09185 \pm 0.00137 , \quad d_1^+ = -1.48536 \pm 0.03995 , \quad (37)$$

$$d_2^+ = 3.24559 \pm 0.3619 , \quad d_3^+ = -3.65645 \pm 0.9676 . \quad (38)$$

For negative z we retain the result of the three-term, asymptotic fit from [6]

$$d_0^- \equiv 1 , \quad d_1^- = 0.273651 \pm 0.002933 , \quad d_2^- = 0.0036058 \pm 0.004875 . \quad (39)$$

The approximations for the small z and asymptotic expansions, described above, overlap for both $z > 0$ and $z < 0$ in large z -ranges. For $z > 0$ we change from the small z to the asymptotic fits at $z = 1.85$, for $z < 0$ at $z = -2.0$. Now everything is fixed and we can calculate the coefficients of the leading asymptotic terms of $f_f(z)$ from Eqs. (59) and (62) of Ref. [6]. We find

$$c_0^+ = 0.417756382 \pm 0.00656 , \quad c_0^- = 0.240933670 \pm 0.00913 . \quad (40)$$

From the last equation one obtains a new estimate of the universal ratio

$$\frac{A^+}{A^-} = 1.734 \pm 0.075 . \quad (41)$$

The new result for the ratio is slightly lower than the result 1.842 ± 0.042 from [6] but still in agreement within the error bars.

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